New Theory of Gravitation

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§1. Introduction

In the General relativity, a metric is used as mathematical expression of the gravity. However, the metric does not resemble gravity. It will be a local inertia coordinate to be good for expression of the gravity. We define ' point-coordinate-systems' as a mathematical expression of the local inertia coordinate. The way of a new gravity theory opened out hereby. On the other hand, we define ' light-cone' . A new mathematical model of space-time is made by this ' point-coordinate-systems' and ' light-cone' . An interesting vector Ai appears when we define a light-ray on this model. This Ai will behave like a vector potential of electromagnetism.

§2. Description of Necessary Mathematics.

In this chapter, because we generally deal with a N-space , the subscripts $i, j, k, l, m, n, ...$ are assumed to take the values $1, 2, 3, ..., N$. We easily write (x^i) the coordinates $(x^1, x^2, ..., x^N)$. A symbol δ^i_j and a symbol δ_{ij} are the $Kronecker's\ delta$.

2.1 Tensors

In this paper, the definition of the tensor followed the reference[1]. We easily introduce it here.

The definition of a tensor of type (m, n) is the following. We describe it by using the example. Let us consider a set of real functions T^{ij}_{klm} in the N-space consisted of N^5 elements. It is said that the set T^{ij}_{klm} is a tensor of type (2,3), if they transform on change of coordinates $(x^i) \rightarrow (\bar{x}^i)$, according to the equations

$$\bar{T}^{op}_{qrs} = \frac{\partial \bar{x}^o}{\partial x^i}\frac{\partial \bar{x}^p}{\partial x^j}\frac{\partial x^k}{\partial \bar{x}^q}\frac{\partial x^l}{\partial \bar{x}^r}\frac{\partial x^m}{\partial \bar{x}^s}T^{ij}_{klm}. \quad (2.1.1)$$

Here, $\bar{T}^{op}_{qrs}$ is defined on coordinates $(\bar{x}^i)$.

A covariant vector A_i is a tensor of type (0,1) because it transform as follows.

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j. \quad (2.1.2)$$

A contravariant vector A^i is a tensor of type (1,0) because it transform as follows.

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j. \quad (2.1.3)$$

2.2 Point-coordinate-systems and coefficients of connection.

Let us consider a point P in the N-space and a neighborhood U_P of P. In U_P, we give a coordinate (z^i) whose origin is P. The (z^i) is called a point-coordinate of P in this paper. If the point-coordinate (z^i) is given to each point in the N-space, they are called a point-coordinate-system in this paper. By using the point-coordinate-system (z^i) , we define the expression ${}^z\Gamma^i_{jk}$ as follows.

$${}^z\Gamma^i_{jk}(P) = \frac{\partial x^i}{\partial z^l} \frac{\partial^2 z^l}{\partial x^j \partial x^k}. \quad (2.2.1)$$

Here, this partial derivatives are evaluated at the origin of (z^i) of P. In this paper, ${}^z\Gamma^i_{jk}$ are called the coefficients of connection defined by the point-coordinate-system (z^i) .

2.3 Covariant derivatives

In this section , we define the covariant derivative of tensor by using the point-coordinate-system (z^i) . These methods are extremely effective for our purpose.

Let us consider a covariant vector E_i and $\bar{E}_i$ defined by the equations

$$\bar{E}_i = \frac{\partial x^j}{\partial z^i} E_j. \quad (2.3.1)$$

It is eazy to prove the following.

$$\frac{\partial z^k}{\partial x^i} \frac{\partial z^l}{\partial x^j} \frac{\partial \bar{E}_k}{\partial z^l} = \frac{\partial E_i}{\partial x^j} - {}^z\Gamma^l_{ij} E_l. \quad (2.3.2)$$

Here, $\partial \bar{E}_k / \partial z^l$ are evaluated at the origin of (z^i) . The expression ${}^z\nabla_j E_i$ is defined by the left-hand side or the right-hand side of (2.3.2). We can prove that ${}^z\nabla_j E_i$ is a tensor of type (0,2). ${}^z\nabla_j E_i$ is called the covariant derivative of E_i concerning ${}^z\Gamma^i_{jk}$ in this paper.

Let us consider a contravariant vector F^i and $\bar{F}^i$ defined by the equations

$$\bar{F}^i = \frac{\partial z^i}{\partial x^j} F^j. \quad (2.3.3)$$

It is eazy to prove the following.

$$\frac{\partial z^k}{\partial x^j}\frac{\partial x^i}{\partial z^l}\frac{\partial \bar{F}^l}{\partial z^k} = \frac{\partial F^i}{\partial x^j} + {}^z\Gamma^i_{jl} F^l. \quad (2.3.4)$$

Here, $\partial \bar{F}^l / \partial z^k$ are evaluated at the origin of (z^i) . The expression ${}^z\nabla_j F^i$ is defined by the left-hand side or the right-hand side of (2.3.4). We can prove that ${}^z\nabla_j F^i$ is a tensor of type (1,1). ${}^z\nabla_j F^i$ is called the covariant derivative of F^i concerning ${}^z\Gamma^i_{jk}$ in this paper.

Similarly in case of other tensors, we can define its covariant derivatives. Let f be a scalar. Let g_{ij} be a tensor of type (0,2). Then, we have the definitions as follows.

$${}^z\nabla_i f = \partial_i f. \quad (2.3.5)$$

$${}^z\nabla_k g_{ij} = \partial_k g_{ij} - {}^z\Gamma^p_{ki} g_{pj} - {}^z\Gamma^p_{kj} g_{ip}. \quad (2.3.6)$$

We can prove that ${}^z\nabla_i f$ is a tensor of type (0,1) and ${}^z\nabla_k g_{ij}$ is a tensor of type (0,3).

Let A_i and B_i be two tensor of type (0,1). Let E_{ij} be a tensor of type (0,2). Let g^{ij} be a tensor of type (2,0). Then, we can prove the following.

$${}^z\nabla_k (A_i + B_i) = {}^z\nabla_k A_i + {}^z\nabla_k B_i.$$

$${}^z\nabla_k (g_{ij} v^j v^j) = ({}^z\nabla_k g_{ij}) v^i v^j + g_{ij} ({}^z\nabla_k v^i) v^j + g_{ij} v^i ({}^z\nabla_k v^j).$$

$${}^z\nabla_k (f E_{ij}) = ({}^z\nabla_k f) E_{ij} + f ({}^z\nabla_k E_{ij}).$$

$${}^z\nabla_k (g^{ij} A_j) = ({}^z\nabla_k g^{ij}) A_j + g^{ij} ({}^z\nabla_k A_j).$$

These equations can be extended to general laws.

2.4 The equation ${}^z[x^i/t] = 0$.

Let us suppose that the coefficients of connection ${}^z\Gamma^i_{jk}$ and a curve $x^i(t)$ are given in the N-space. We define the expression ${}^z[x^i/t]$ as follows.

$${}^z[x^i/t] = \frac{dv^i}{dt} + {}^z\Gamma^i_{jk} v^j v^k \quad , \quad v^i = \frac{dx^i}{dt}. \quad (2.4.1)$$

The ${}^z[x^i/t]$ are vectors on the curve $x^i(t)$.

Let $x^i(t)$ be the solution of ${}^z[x^i/t] = 0$. If we change the parameter from t to s , then $x^i(s)$ generally is not the solution of ${}^z[x^i/s] = 0$. Therefore, t is the special parameter of this curve. The t is called a orthonormal parameter of this curve in this paper.

Let t be the orthonormal parameter. Let c be an arbitrary constant. Then ct is also the orthonormal parameter. In addition, if s is an arbitrary orthonormal parameter, then we have $s = \bar{c}t$ as follows. Here, $\bar{c}$ is a certain constant. By using (3) of section 2.5,

$$ {}^z[x^i/s] = \left(\frac{dt}{ds}\right)^2 {}^z[x^i/t] + \frac{d^2t}{ds^2}v^i = 0 \quad , \quad v^i = \frac{dx^i}{dt}. \quad (2.4.2) $$

By (2.4.2), we obtain $d^2t/ds^2 = 0$,i.e., $s = \bar{c}t$.

In (2,4,1), the vector v^i is defined only on the curve, however we virtually can extend v^i to neighborhood of the curve. Then we can write ${}^z[x^i/t]$ as follows.

$$ {}^z[x^i/t] = \left(\frac{\partial v^i}{\partial x^k} + {}^z\Gamma^i_{jk}v^j\right)v^k = ({}^z\nabla_k v^i)v^k. \quad (2.4.3) $$

Lemma 2.4.1

Suppose that the coefficient of connection ${}^z\Gamma^i_{jk}$ and the metric tensor g_{ij} are given in the N-space. Let the curve $x^i(t)$ be a solution of ${}^z[x^i/t] = 0$. Let a parameter s be the arc-length measured with g_{ij} along this curve . Then, we obtain the following.

$$ \frac{d^2s}{dt^2} - \frac{1}{2}({}^z\nabla_k g_{ij})V^iV^jV^k\left(\frac{ds}{dt}\right)^2 = 0 \quad , \quad V^i = \frac{dx^i}{ds}. \quad (1) $$

$$ \frac{d^2t}{ds^2} + \frac{1}{2}({}^z\nabla_k g_{ij})V^iV^jV^k\frac{dt}{ds} = 0. \quad (2) $$

(proof) By (3) of section 2.5,

$$ {}^z[x^i/t] = \left(\frac{ds}{dt}\right)^2 {}^z[x^i/s] + \frac{d^2s}{dt^2}V^i = 0. $$

Multiplication by $g_{ij}V^j$ gives

$$ \left(\frac{ds}{dt}\right)^2 g_{ij}\ {}^z[x^i/s]V^j + \frac{d^2s}{dt^2} = 0. \quad (3) $$

By $g_{ij}V^iV^j = 1$, we have

$$ 0 = {}^z\nabla_k(g_{ij}V^iV^j)V^k = ({}^z\nabla_k g_{ij})V^iV^jV^k + 2g_{ij}({}^z\nabla_k V^i)V^kV^j. $$

Because $({}^{z}\nabla_k V^i)V^k = {}^{z}[x^i/s]$, we have

$$({}^{z}\nabla_k g_{ij})V^i V^j V^k = -2g_{ij}\ {}^{z}[x^i/s]V^j. \quad (4)$$

By setting (4) to (3), we obtain the equation (1). Lastly, by using (1) of section 2.5 to (1), we obtain the equation (2). □

2.5 Formulae.

In this section, we give the formulae using in this paper. We can prove these formulae by the simple calculation.

Suppose that t is some function of s , then we have

$$\frac{d^2 s}{dt^2} = -\Big(\frac{ds}{dt}\Big)^3 \frac{d^2 t}{ds^2}. \quad (1)$$

Suppose that $(x^i), (y^i)$ are two coordinates in the N-space and $x^i(t)$ is a curve in the N-space , then we have

$$\frac{d^2 y^i}{dt^2} = \frac{\partial y^i}{\partial x^n}\Big(\frac{d^2 x^n}{dt^2} + \frac{\partial x^n}{\partial y^l}\frac{\partial^2 y^l}{\partial x^j \partial x^k}\frac{dx^j}{dt}\frac{dx^k}{dt}\Big). \quad (2)$$

Suppose that a coefficient of connection ${}^{a}\Gamma^i_{jk}$ and a curve $x^i(t)$ are given in the N-space. Let s be an arbitrary parameter of this curve. Then we have

$${}^{a}[x^i/t] = \Big(\frac{ds}{dt}\Big)^2\ {}^{a}[x^i/s] + \frac{d^2 s}{dt^2}\frac{dx^i}{ds}. \quad (3)$$

§3. Mathematical Model of Space-Time.

In the first, let us suppose that our space-time consist of four dimensions. Suppose that the subscripts $i, j, k, l, m, n, ..., z$ take the values $1, 2, 3, 4$ and the subscripts $\alpha, \beta, ..., \omega$ take the values $0, 1, 2, 3, 4$.

3.1 Point-coordinate-systems expressing inertia and equations of free-fall.

Let us construct the space-time in the 4-space. First, we consider a free-fall of the material-point. Here, suppose that the curve of free-fall is irrelevant to its mass. At each point of the space-time, we can image the inertial frame of reference. Then, let us suppose that a certain point-coordinate-system (y^i) expresses the inertial frame of reference.

Let a curve $x^i(\tau)$ be the free-fall of the material-point. Here, τ is the proper-time. Let P be some point on this curve. If we see this curve in the point-coordinate (y^i) of P , then we will have

$$\frac{d^2y^i}{d\tau^2} = 0.$$

By using (2) of section 2.5, we have

$$\frac{d^2y^i}{d\tau^2} = \frac{\partial y^i}{\partial x^n}\Big(\frac{d^2x^n}{d\tau^2} + \frac{\partial x^n}{\partial y^l}\frac{\partial^2 y^l}{\partial x^j \partial x^k}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau}\Big) = 0. \quad (3.1.1)$$

The equation (3.1.1) is identical to

$$^{y}[x^i/\tau] = 0. \quad (3.1.2)$$

The (3.1.2) is the equation of the free-fall and the proper-time τ is the orthonormal parameter of this curve.

3.2 Light-cones and equations of light-ray.

We define the matrix B_{ij} as follows.

$$B_{11} = B_{22} = B_{33} = -1 \ \ , \ \ B_{44} = 1 \ \ , \ \ B_{ij} = 0 \ \ if \ i \neq j. \quad (3.2.1)$$

Let P be an arbitrary point in the 4-space. Suppose that the light-cone $G_{ij}(P)$ of P has some following features.

$$G_{ij}(P) = G_{ji}(P). \quad (3.2.2)$$

If a vector v^i grown from P is the direction of the light-ray starting from P , then

$$G_{ij}(P)v^iv^j = 0. \quad (3.2.3)$$

The light-cone G_{ij} is the tensor of type (0,2). Let λ be an arbitrary scalar. If G_{ij} is the light-cone , then λG_{ij} is also the light-cone of the same light-wave. Additionally, a non-singular matrix S^i_j exists as follows.

$$S^k_i S^l_j G_{kl} = B_{ij}. \quad (3.2.4)$$

Already, we gave the equation of free-fall of the material-point. Similarly, the equation of the light-ray $x^i(\tau)$ is also given by (3.1.2). On the other hand, the light-ray has to meet the equation (3.2.3) at all points. Therefore, we have

$$0 = \frac{d}{d\tau}(G_{ij}v^i v^j) = {}^y\nabla_k(G_{ij}v^i v^j)v^k$$

$$= ({}^y\nabla_k G_{ij})v^i v^j v^k + 2G_{ij}({}^y\nabla_k v^i)v^k v^j \quad , \quad v^i = \frac{dx^i}{d\tau}. \quad (3.2.5)$$

By setting

$$({}^y\nabla_k v^i)v^k = {}^y[x^i/\tau] = 0,$$

we obtain

$$({}^y\nabla_k G_{ij})v^i v^j v^k = 0. \quad (3.2.6)$$

The equation (3.2.6) has to apply to all the light-rays starting from P. Therefore, the polynomial $({}^y\nabla_k G_{ij})X^i X^j X^k$ can just be divided by the polynomial $G_{ij}X^i X^j$, because $G_{ij}X^i X^j$ is irreducible by Lemma 3.2.1 (→ reference[2]) . Therefore $2A_i$ exists as follows.

$${}^y\nabla_k G_{ij} X^i X^j X^k = (2A_i X^i)(G_{jk} X^j X^k). \quad (3.2.7)$$

Now, we pay attention to the A_i . Let us change the light-cone from G_{ij} to $\bar{G}_{ij} = \lambda G_{ij}$. By the equation

$${}^y\nabla_k \bar{G}_{ij} = (\partial_k\lambda)G_{ij} + \lambda\, {}^y\nabla_k G_{ij}, \quad (3.2.8)$$

we have

$${}^y\nabla_k \bar{G}_{ij} X^i X^j X^k = (\partial_k\lambda)G_{ij} X^i X^j X^k + \lambda\, {}^y\nabla_k G_{ij} X^i X^j X^k. \quad (3.2.9)$$

By setting (3.2.7) to (3.2.9), we have

$${}^y\nabla_k \bar{G}_{ij} X^i X^j X^k = \{(\partial_k\lambda)G_{ij} + 2\lambda A_k G_{ij}\} X^i X^j X^k$$

$$= 2\Big(\frac{1}{2\lambda}\partial_k\lambda + A_k\Big)\bar{G}_{ij} X^i X^j X^k. \quad (3.2.10)$$

By the (3.2.10), we obtain

$$\bar{A}_k = A_k + \partial_k \log\sqrt{\lambda}. \quad (3.2.11)$$

Here, $\bar{A}_i$ is corresponding to $\bar{G}_{ij}$. By the equation (3.2.11), it seems that A_i is the vector potential of electromagnetism.

Lemma 3.2.1

If G_{ij} is a light-cone, the polynomial $G_{ij}X^iX^j$ is irreducible.

(proof) We will lead a contradiction from the supposition which $G_{ij}X^iX^j$ is reducible. By a certain non-singular matrix S_i^j , we have

$$B_{ij} = S_i^k S_j^l G_{kl}. \quad (1)$$

If $G_{ij}X^iX^j$ is reducible, a_i and b_i exist as follows.

$$G_{ij}X^iX^j = a_iX^ib_jX^j. \quad (2)$$

Therefore we have

$$G_{ij} = \frac{1}{2}(a_ib_j + a_jb_i). \quad (3)$$

By using (1) and (3), we have

$$B_{ij} = \frac{1}{2}S_i^kS_j^l(a_kb_l + a_lb_k) = \frac{1}{2}(\bar{a}_i\bar{b}_j + \bar{a}_j\bar{b}_i). \quad (4)$$

Here,

$$\bar{a}_i = S_i^p a_p\ ,\ \bar{b}_i = S_i^p b_p. \quad (5)$$

In the special case of (4), we have

$$-1 = B_{11} = \bar{a}_1\bar{b}_1 \quad , \quad -1 = B_{22} = \bar{a}_2\bar{b}_2. \quad (6)$$

Therefore we have

$$\bar{b}_1 = -\frac{1}{\bar{a}_1} \quad , \quad \bar{b}_2 = -\frac{1}{\bar{a}_2}. \quad (7)$$

Similarly by using (4), we have

$$0 = B_{12} = \frac{1}{2}(\bar{a}_1\bar{b}_2 + \bar{a}_2\bar{b}_1). \quad (8)$$

By setting (7) to (8), we have

$$0 = -\frac{1}{2}\Big(\frac{\bar{a}_1}{\bar{a}_2} + \frac{\bar{a}_2}{\bar{a}_1}\Big). \quad (9)$$

Multiplication by $\bar{a}_1\bar{a}_2$ to (9), we have

$$0 = \bar{a}_1\bar{a}_1 + \bar{a}_2\bar{a}_2. \quad (10)$$

We obtain $\bar{a}_1 = \bar{a}_2 = 0$ by (10), however these results contradict (6). □

3.3 Space-time-potential and guage transformations.

Suppose that the light-cone G_{ij} and the point-coordinate-system (y^i) expressing the inertial frame of reference are given in the 4-space. Let $x^i(\tau)$ be the curve of free-fall of the material-point. Let s be the arc-length measured with the metric G_{ij} along this curve, i.e.,

$$ds^2 = G_{ij}dx^i dx^j. \quad (3.3.1)$$

According to Lemma 2.4.1

$$\frac{d^2\tau}{ds^2} + \frac{1}{2}({}^y\nabla_k G_{ij})V^iV^jV^k\frac{d\tau}{ds} = 0 \ \ , \ \ V^i = \frac{dx^i}{ds}. \quad (3.3.2)$$

On the other hand, according to the section 3.2 ,

$$({}^y\nabla_k G_{ij})V^iV^jV^k = 2(A_kV^k)(G_{ij}V^iV^j). \quad (3.3.3)$$

Because $G_{ij}V^iV^j = 1$, we obtain

$$\frac{d^2\tau}{ds^2} + (A_kV^k)\frac{d\tau}{ds} = 0. \quad (3.3.4)$$

Let P, Q be two point on the $x^i(\tau)$. We consider

$$\zeta(P) = -\int_Q^P A_i dx^i + C. \quad (3.3.5)$$

Here, C is a constant. If τ is defined as

$$d\tau = \exp(\zeta)ds, \quad (3.3.6)$$

then

$$\frac{d^2\tau}{ds^2} = \exp(\zeta)\frac{d\zeta}{ds} = -\exp(\zeta)A_i\frac{dx^i}{ds}. \quad (3.3.7)$$

The equation (3.3.7) shows that τ is the solution of the equation (3.3.4). In this paper, ζ is called a space-time-potential.

By (3.3.6),

$$d\tau^2 = \exp(2\zeta)G_{ij}dx^i dx^j. \quad (3.3.8)$$

We hope to deal with $\exp(2\zeta)G_{ij}$ as the metric , however ζ is not a function in the 4-space (x^i) . Then, let us extend the space-time to a 5-space (x^λ) , and let us consider $x^0 = \zeta$. We define a new metric $g_{\lambda\mu}$ in the 5-space (x^λ) as follows.

$$g_{ij} = \exp(2x^0)G_{ij}(x^1, ..., x^4) \ , \ g_{\lambda 0} = g_{0\lambda} = 0. \quad (3.3.9)$$

According to the definitions, the curve $x^i(\tau)$ is written $x^\lambda(\tau)$ in the 5-space (x^λ). Let dx^λ be a line element on this curve. Then,

$$dx^0 = d\zeta = -A_i dx^i, \quad (3.3.10)$$

i.e.,

$$dx^0 + A_i dx^i = 0. \quad (3.3.11)$$

If we define $A_0 = 1$ as a fifth element of A_i , then we can write (3.3.11) as follows.

$$A_\lambda dx^\lambda = 0. \quad (3.3.12)$$

In this paper, transformations appeared by $G_{ij} \to \lambda G_{ij}$ are called a gauge transformation. As an example, we have

$$A_i \to A_i + \partial_i \eta \ \ , \ \ \eta = \log \sqrt{\lambda}. \quad (3.3.13)$$

How does the space-time-potential of the curve transform by the gauge transformation ? Let $\bar{\zeta}$ be a space-time-potential of the new gauge. According to the definitions,

$$d\bar{\zeta} = -(A_i + \partial_i \eta) dx^i \ \ , \ \ \bar{\zeta}(P) = -\int_Q^P d\bar{\zeta} + C. \quad (3.3.14)$$

Here, Q and C are not fixed. Then, let us suppose that the proper-time does not vary by the gauge transformation. That is,

$$d\tau^2 = \exp(2\zeta) G_{ij} dx^i dx^j = \exp(2\bar{\zeta}) \lambda G_{ij} dx^i dx^j$$

$$= \exp(2\bar{\zeta} + 2\eta) G_{ij} dx^i dx^j. \quad (3.3.15)$$

Therefore

$$\zeta(P) = \bar{\zeta}(P) + \eta(P). \quad (3.3.16)$$

Now, we consider the transformation of coordinates as follows.

$$\bar{x}^0 = x^0 - \eta(x^1, ..., x^4) \ , \ \bar{x}^i = x^i. \quad (3.3.17)$$

By (3.3.17), A_λ transform as follows.

$$\bar{A}_0 = \frac{\partial x^0}{\partial \bar{x}^0} A_0 + \frac{\partial x^j}{\partial \bar{x}^0} A_j = 1 + \delta_0^j A_j = 1, \quad (3.3.18)$$

$$\bar{A}_i = \frac{\partial x^0}{\partial \bar{x}^i} A_0 + \frac{\partial x^j}{\partial \bar{x}^i} A_j = \partial_i \eta + \delta_i^j A_j = A_i + \partial_i \eta. \quad (3.3.19)$$

Generally by using (3.3.17), a symmetric tensor $c_{\lambda\mu}$ of type (0,2) transform as follows.

$$\bar{c}_{ij} = c_{ij} + \partial_i \eta c_{0j} + \partial_j \eta c_{0i} + \partial_i \eta \partial_j \eta c_{00} \ ,$$

$$\bar{c}_{0j} = c_{0j} + \partial_j \eta c_{00} \ , \ \bar{c}_{00} = c_{00}. \quad (3.3.20)$$

In the case of $g_{\lambda\mu}$, we have

$$\bar{g}_{ij} = g_{ij} \quad , \quad \bar{g}_{\lambda 0} = \bar{g}_{0\lambda} = 0. \quad (3.3.21)$$

3.4 Metrics of 5-space.

The metric $g_{\lambda\nu}$ defined in section 3.3 has not a inverse matrix. If $g_{\lambda\nu}$ has a inverse matrix $g^{\lambda\nu}$ then $g^{\lambda\nu} g_{\nu\mu} = \delta^{\lambda}_{\mu}$. In the case of $\lambda = \mu = 0$,

$$0 = g^{0\nu} g_{\nu 0} = \delta^0_0 = 1.$$

This is a contradiction. Therefore, $g_{\lambda\nu}$ is abnormal as the metric of the 5-space. Let us define a normal metric $h_{\lambda\mu}$ extended $g_{\lambda\mu}$.

If a vector V^λ grown from a point P is $A_\lambda(P) V^\lambda = 0$ then we wish

$$h_{\lambda\mu}(P) V^\lambda V^\mu = g_{\lambda\mu}(P) V^\lambda V^\mu. \quad (3.4.1)$$

Therefore, the polynomial

$$(h_{\lambda\mu} - g_{\lambda\mu}) X^\lambda X^\mu \quad (3.4.2)$$

can just be divided by the polynomial $A_\mu X^\mu$. We can find out a_λ as follows.

$$(h_{\lambda\mu} - g_{\lambda\mu}) X^\lambda X^\mu = (a_\lambda X^\lambda)(A_\mu X^\mu). \quad (3.4.3)$$

As a result, we obtain

$$h_{\lambda\mu} = g_{\lambda\mu} + \frac{1}{2}(a_\lambda A_\mu + a_\mu A_\lambda). \quad (3.4.4)$$

By (3.3.20), the metric $h_{\lambda\mu}$ transforms as follows.

$$\bar{h}_{ij} = h_{ij} + \partial_i \eta h_{0j} + \partial_j \eta h_{0i} + \partial_i \eta \partial_j \eta h_{00}, \quad (3.4.5)$$

$$\bar{h}_{0j} = h_{0j} + \partial_j \eta h_{00} \ , \ \bar{h}_{00} = h_{00}. \quad (3.4.6)$$

In (3.4.6), we know that h_{0j}/h_{00} has the same transformation as A_i . Therefore, let us define the following.

$$h_{0j} = h_{00} A_j. \quad (3.4.7)$$

By using (3.4.4),

$$h_{00} = a_0. \quad (3.4.8)$$

By using (3.4.7) and (3.4.8),

$$h_{0j} = a_0 A_j. \quad (3.4.9)$$

On the other hand, by using (3.4.4)

$$h_{0j} = \frac{1}{2}(a_0 A_j + a_j). \quad (3.4.10)$$

By using (3.4.10) and (3.4.9)

$$a_j = a_0 A_j.$$

On the other hand $a_0 = a_0 A_0$, therefore $a_\lambda = a_0 A_\lambda$. As a result, we obtain

$$h_{\lambda\mu} = g_{\lambda\mu} + a_0 A_\lambda A_\mu. \quad (3.4.11)$$

Lastly, we have to decide a_0. Let us consider $dx^\lambda = (dx^0, 0, 0, 0, 0)$. The length of dx^λ is

$$dl^2 = h_{\lambda\mu} dx^\lambda dx^\mu = h_{00} dx^0 dx^0 = a_0 dx^0 dx^0. \quad (3.4.12)$$

We will expect $dl^2 = dx^0 dx^0$, i.e., $a_0 = 1$. We obtain

$$h_{\lambda\mu} = \exp(2x^0) G_{\lambda\mu} + A_\lambda A_\mu. \quad (3.4.13)$$

If we disregard $\exp(2x^0)$, $h_{\lambda\mu}$ is same as the $Kaluza's\ metric$.

The $h_{\lambda\mu}$ has a inverse matrix $h^{\lambda\mu}$ as follows.

$$h^{ij} = g^{ij} \ , \ h^{i0} = h^{0i} = -g^{ij} A_j \ ,$$

$$h^{00} = g^{ij} A_i A_j + 1 \ , \ g^{ij} = \exp(-2x^0) G^{ij} \ .$$

References

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