New Theory of Gravitation

Mitsuru Watanabe
Shizuoka-ken Japan

§1. Introduction
In the General relativity, a metric is used as mathematical expression of the gravity. However, the metric does not resemble gravity. It will be a local inertia coordinate to be good for expression of the gravity. We define 'point-coordinate-systems' as a mathematical expression of the local inertia coordinate. The way of a new gravity theory opened out hereby. On the other hand, we define 'light-cone'. A new mathematical model of space-time is made by this 'point-coordinate-systems' and 'light-cone'. An interesting vector $A_i$ appears when we define a light-ray on this model. This $A_i$ will behave like a vector potential of electromagnetism.

§2. Description of Necessary Mathematics.
In this chapter, because we generally deal with a $N$-space, the subscripts $i,j,k,l,m,n,...$ are assumed to take the values $1,2,3,...,N$. We easily write $(x^i)$ the coordinates $(x^1,x^2,...,x^N)$. A symbol $\delta^i_j$ and a symbol $\delta_{ij}$ are the Kronecker's delta.

2.1 Tensors
In this paper, the definition of the tensor followed the reference[1]. We easily introduce it here.

The definition of a tensor of type $(m,n)$ is the following. We describe it by using the example. Let us consider a set of real functions $T_{ijkl}^{mnop}$ in the $N$-space consisted of $N^5$ elements. It is said that the set $T_{ijkl}^{mnop}$ is a tensor of type $(2,3)$, if they transform on change of coordinates $(x^i) \rightarrow (\tilde{x}^i)$, according to the equations

$$T_{qrst}^{nop} = \frac{\partial \tilde{x}^a}{\partial x^q} \frac{\partial \tilde{x}^p}{\partial x^r} \frac{\partial \tilde{x}^k}{\partial x^s} \frac{\partial \tilde{x}^l}{\partial x^t} T_{ijkl}^{mnop} \quad (2.1.1)$$

Here, $T_{qrst}^{nop}$ is defined on coordinates $(\tilde{x}^i)$. 

A covariant vector $A_i$ is a tensor of type (0,1) because it transforms as follows.

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j. \quad (2.1.2)$$

A contravariant vector $A^i$ is a tensor of type (1,0) because it transforms as follows.

$$\bar{A}^i = \frac{\partial \bar{z}^l}{\partial x^j} A^j. \quad (2.1.3)$$

### 2.2 Point-coordinate-systems and coefficients of connection.

Let us consider a point $P$ in the $N$-space and a neighborhood $U_P$ of $P$. In $U_P$, we give a coordinate $(z^i)$ whose origin is $P$. The $(z^i)$ is called a point-coordinate of $P$ in this paper. If the point-coordinate $(z^i)$ is given to each point in the $N$-space, they are called a point-coordinate-system in this paper. By using the point-coordinate-system $(z^i)$, we define the expression $z^j_{ij}$ as follows.

$$z^j_{ij}(P) = \frac{\partial z^j}{\partial x^i} \frac{\partial z^l}{\partial x^j} A^l. \quad (2.2.1)$$

Here, this partial derivatives are evaluated at the origin of $(z^i)$ of $P$. In this paper, $z^j_{ij}$ are called the coefficients of connection defined by the point-coordinate-system $(z^i)$.

### 2.3 Covariant derivatives

In this section, we define the covariant derivative of tensor by using the point-coordinate-system $(z^i)$. These methods are extremely effective for our purpose.

Let us consider a covariant vector $E_i$ and $\bar{E}_i$ defined by the equations

$$\bar{E}_i = \frac{\partial x^j}{\partial \bar{z}^i} E_j. \quad (2.3.1)$$

It is easy to prove the following.

$$\frac{\partial z^k}{\partial x^l} \frac{\partial z^j}{\partial x^i} \frac{\partial E_k}{\partial \bar{x}^j} = \frac{\partial E_l}{\partial \bar{x}^j} - z^j_{ij} E_l. \quad (2.3.2)$$

Here, $\partial E_k / \partial z^j$ are evaluated at the origin of $(z^i)$. The expression $z^j_{ij} E_i$ is defined by the left-hand side or the right-hand side of (2.3.2). We can prove that $z^j_{ij} E_i$ is a tensor of type (0,2). $z^j_{ij} E_i$ is called the covariant derivative of $E_i$ concerning $z^j_{ij}$ in this paper.
Let us consider a contravariant vector $F^i$ and $\tilde{F}^i$ defined by the equations

$$\tilde{F}^i = \frac{\partial z^i}{\partial x^j} F^j. \quad (2.3.3)$$

It is easy to prove the following.

$$\frac{\partial z^k}{\partial x^j} \frac{\partial x^i}{\partial z^k} = \frac{\partial F^i}{\partial x^j} + z^i \Gamma^j_{\mu} F^\mu. \quad (2.3.4)$$

Here, $\partial \tilde{F}^i / \partial z^k$ are evaluated at the origin of $(z^i)$. The expression $z^i \nabla_j F^i$ is defined by the left-hand side or the right-hand side of (2.3.4). We can prove that $z^i \nabla_j F^i$ is a tensor of type (1,1). $z^i \nabla_j F^i$ is called the covariant derivative of $F^i$ concerning $z^i \Gamma^j_{\mu}$ in this paper.

Similarly in case of other tensors, we can define its covariant derivatives. Let $f$ be a scalar. Let $g_{ij}$ be a tensor of type $(0,2)$. Then, we have the definitions as follows.

$$z^i \nabla_j f = \partial_i f. \quad (2.3.5)$$

$$z^i \nabla_k g_{ij} = \partial_k g_{ij} - z^p \Gamma^i_{jk} g_{pj} - z^p \Gamma^j_{kp} g_{ip}. \quad (2.3.6)$$

We can prove that $z^i \nabla_j f$ is a tensor of type $(0,1)$ and $z^i \nabla_k g_{ij}$ is a tensor of type $(0,3)$.

Let $A_i$ and $B_i$ be two tensor of type $(0,1)$. Let $E_{ij}$ be a tensor of type $(0,2)$. Let $g^{ij}$ be a tensor of type $(2,0)$. Then, we can prove the following.

$$z^i \nabla_k (A_i + B_i) = z^i \nabla_k A_i + z^i \nabla_k B_i.$$  

$$z^i \nabla_k (g_{ij} v^j v^i) = (z^i \nabla_k g_{ij}) v^j v^i + g_{ij} (z^i \nabla_k v^j) v^i + g_{ij} v^j (z^i \nabla_k v^i).$$

$$z^i \nabla_k (f E_{ij}) = (z^i \nabla_k f) E_{ij} + f (z^i \nabla_k E_{ij}).$$

$$z^i \nabla_k (g^{ij} A_j) = (z^i \nabla_k g^{ij}) A_j + g^{ij} (z^i \nabla_k A_j).$$

These equations can be extended to general laws.

2.4 The equation $z^i [x^i / t] = 0$.

Let us suppose that the coefficients of connection $z^i \Gamma^j_{\mu}$ and a curve $x^i(t)$ are given in the $N$-space. We define the expression $z^i [x^i / t]$ as follows.

$$z^i [x^i / t] = \frac{dv^i}{dt} + z^i \Gamma^j_{jk} v^j v^k, \quad v^i = \frac{dx^i}{dt}. \quad (2.4.1)$$
The $z[x^i/t]$ are vectors on the curve $x^i(t)$.

Let $x^i(t)$ be the solution of $z[x^i/t] = 0$. If we change the parameter from $t$ to $s$, then $x^i(s)$ generally is not the solution of $z[x^i/s] = 0$. Therefore, $t$ is the special parameter of this curve. The $t$ is called a orthonormal parameter of this curve in this paper.

Let $t$ be the orthonormal parameter. Let $c$ be an arbitrary constant. Then $ct$ is also the orthonormal parameter. In addition, if $s$ is an arbitrary orthonormal parameter, then we have $s = \hat{c}t$ as follows. Here, $\hat{c}$ is a certain constant. By using (3) of section 2.5,

$$z[x^i/s] = \left(\frac{dt}{ds}\right)^2 z[x^i/t] + \frac{d^2t}{ds^2} v^i = 0, \quad v^i = \frac{dx^i}{dt}. \quad (2.4.2)$$

By (2.4.2), we obtain $d^2t/ds^2 = 0$, i.e., $s = \hat{c}t$.

In (2.4.1), the vector $v^i$ is defined only on the curve, however we virtually can extend $v^i$ to neighborhood of the curve. Then we can write $z[x^i/t]$ as follows.

$$z[x^i/t] = (\frac{\partial v^i}{\partial x^k} + z^i_{jk} v^j) v^k = (z_k v^i) v^k. \quad (2.4.3)$$

Lemma 2.4.1

Suppose that the coefficient of connection $z^i_{jk}$ and the metric tensor $g_{ij}$ are given in the $N$-space. Let the curve $x^i(t)$ be a solution of $z[x^i/t] = 0$. Let a parameter $s$ be the arc-length measured with $g_{ij}$ along this curve. Then, we obtain the following.

$$\frac{d^2s}{dt^2} - \frac{1}{2} (z^i_{k}g_{ij}) V^i V^j V^k \left(\frac{ds}{dt}\right)^2 = 0, \quad V^i = \frac{dx^i}{ds}. \quad (1)$$

$$\frac{d^2t}{ds^2} + \frac{1}{2} (z^i_{k}g_{ij}) V^i V^j V^k \frac{dt}{ds} = 0. \quad (2)$$

(proof) By (3) of section 2.5,

$$z[x^i/t] = \left(\frac{ds}{dt}\right)^2 z[x^i/s] + \frac{d^2s}{dt^2} V^i = 0.$$

Multiplication by $g_{ij} V^j$ gives

$$\left(\frac{ds}{dt}\right)^2 g_{ij} z[x^i/s] V^j + \frac{d^2s}{dt^2} V^i = 0. \quad (3)$$

By $g_{ij} V^i V^j = 1$, we have

$$0 = z^i_k (g_{ij} V^i V^j) V^k = (z^i_k g_{ij}) V^i V^j V^k + 2 g_{ij} (z^i_k V^i) V^k V^j.$$

The $z[x^i/t]$ are vectors on the curve $x^i(t)$. If we change the parameter from $t$ to $s$, then $x^i(s)$ generally is not the solution of $z[x^i/s] = 0$. Therefore, $t$ is the special parameter of this curve. The $t$ is called a orthonormal parameter of this curve in this paper.

Let $t$ be the orthonormal parameter. Let $c$ be an arbitrary constant. Then $ct$ is also the orthonormal parameter. In addition, if $s$ is an arbitrary orthonormal parameter, then we have $s = \hat{c}t$ as follows. Here, $\hat{c}$ is a certain constant. By using (3) of section 2.5,

$$z[x^i/s] = \left(\frac{dt}{ds}\right)^2 z[x^i/t] + \frac{d^2t}{ds^2} v^i = 0, \quad v^i = \frac{dx^i}{dt}. \quad (2.4.2)$$

By (2.4.2), we obtain $d^2t/ds^2 = 0$, i.e., $s = \hat{c}t$.

In (2.4.1), the vector $v^i$ is defined only on the curve, however we virtually can extend $v^i$ to neighborhood of the curve. Then we can write $z[x^i/t]$ as follows.

$$z[x^i/t] = (\frac{\partial v^i}{\partial x^k} + z^i_{jk} v^j) v^k = (z_k v^i) v^k. \quad (2.4.3)$$

Lemma 2.4.1

Suppose that the coefficient of connection $z^i_{jk}$ and the metric tensor $g_{ij}$ are given in the $N$-space. Let the curve $x^i(t)$ be a solution of $z[x^i/t] = 0$. Let a parameter $s$ be the arc-length measured with $g_{ij}$ along this curve. Then, we obtain the following.

$$\frac{d^2s}{dt^2} - \frac{1}{2} (z^i_{k}g_{ij}) V^i V^j V^k \left(\frac{ds}{dt}\right)^2 = 0, \quad V^i = \frac{dx^i}{ds}. \quad (1)$$

$$\frac{d^2t}{ds^2} + \frac{1}{2} (z^i_{k}g_{ij}) V^i V^j V^k \frac{dt}{ds} = 0. \quad (2)$$

(proof) By (3) of section 2.5,

$$z[x^i/t] = \left(\frac{ds}{dt}\right)^2 z[x^i/s] + \frac{d^2s}{dt^2} V^i = 0.$$

Multiplication by $g_{ij} V^j$ gives

$$\left(\frac{ds}{dt}\right)^2 g_{ij} z[x^i/s] V^j + \frac{d^2s}{dt^2} V^i = 0. \quad (3)$$

By $g_{ij} V^i V^j = 1$, we have

$$0 = z^i_k (g_{ij} V^i V^j) V^k = (z^i_k g_{ij}) V^i V^j V^k + 2 g_{ij} (z^i_k V^i) V^k V^j.$$
Because \((\nabla_k V_i)V^k = z[x^i/s]\), we have

\[
(\nabla_k g_{ij})V^i V^j V^k = -2g_{ij} z[x^i/s]V^j. \tag{4}
\]

By setting (4) to (3), we obtain the equation (1). Lastly, by using (1) of section 2.5 to (1), we obtain the equation (2).

2.5 Formulae.

In this section, we give the formulae using in this paper. We can prove these formulae by the simple calculation.

Suppose that \(t\) is some function of \(s\), then we have

\[
\frac{d^2s}{dt^2} = -\left(\frac{ds}{dt}\right)^3 \frac{d^2t}{ds^2}. \tag{1}
\]

Suppose that \((x^i),(y^i)\) are two coordinates in the \(N\)-space and \(x^i(t)\) is a curve in the \(N\)-space, then we have

\[
\frac{d^2y^i}{dt^2} = \frac{\partial y^i}{\partial x^n} \left( \frac{d^2x^n}{dt^2} + \frac{\partial x^n}{\partial y^l} \frac{\partial^2 y^l}{\partial x^j \partial x^k} \frac{dx^j}{dt} \frac{dx^k}{dt} \right). \tag{2}
\]

Suppose that a coefficient of connection \(\Gamma^i_{jk}\) and a curve \(x^i(t)\) are given in the \(N\)-space. Let \(s\) be an arbitrary parameter of this curve. Then we have

\[
a_{[x^i/t]} = \left(\frac{ds}{dt}\right)^2 a_{[x^i/s]} + \frac{d^2s}{dt^2} \frac{dx^i}{ds}. \tag{3}
\]

§3. Mathematical Model of Space-Time.

In the first, let us suppose that our space-time consist of four dimensions. Suppose that the subscripts \(i, j, k, l, m, n, ..., z\) take the values \(1, 2, 3, 4\) and the subscripts \(\alpha, \beta, ..., \omega\) take the values \(0, 1, 2, 3, 4\).

3.1 Point-coordinate-systems expressing inertia and equations of free-fall.

Let us construct the space-time in the 4-space. First, we consider a free-fall of the material-point. Here, suppose that the curve of free-fall is irrelevant to its mass. At each point of the space-time, we can image the inertial frame of reference. Then, let us suppose that a certain point-coordinate-system \((y^i)\) expresses the inertial frame of reference.
Let a curve $x^i(\tau)$ be the free-fall of the material-point. Here, $\tau$ is the proper-time. Let $P$ be some point on this curve. If we see this curve in the point-coordinate $(y^i)$ of $P$, then we will have

$$\frac{d^2 y^i}{d\tau^2} = 0.$$ 

By using (2) of section 2.5, we have

$$\frac{d^2 y^i}{d\tau^2} = \frac{\partial y^i}{\partial x^n} \left( \frac{d^2 x^n}{d\tau^2} + \frac{\partial x^n}{\partial y^l} \frac{\partial y^l}{\partial x^j} \frac{dx^j}{d\tau} \frac{dx^k}{d\tau} \right) = 0. \quad (3.1.1)$$

The equation (3.1.1) is identical to

$$y[x^i/\tau] = 0. \quad (3.1.2)$$

The (3.1.2) is the equation of the free-fall and the proper-time $\tau$ is the orthonormal parameter of this curve.

### 3.2 Light-cones and equations of light-ray.

We define the matrix $B_{ij}$ as follows.

$$B_{11} = B_{22} = B_{33} = -1, \quad B_{44} = 1, \quad B_{ij} = 0 \text{ if } i \neq j. \quad (3.2.1)$$

Let $P$ be an arbitrary point in the 4-space. Suppose that the light-cone $G_{ij}(P)$ of $P$ has some following features.

$$G_{ij}(P) = G_{ji}(P). \quad (3.2.2)$$

If a vector $v^i$ grown from $P$ is the direction of the light-ray starting from $P$, then

$$G_{ij}(P)v^iv^j = 0. \quad (3.2.3)$$

The light-cone $G_{ij}$ is the tensor of type (0,2). Let $\lambda$ be an arbitrary scalar. If $G_{ij}$ is the light-cone, then $\lambda G_{ij}$ is also the light-cone of the same light-wave. Additionally, a non-singular matrix $S^i_j$ exists as follows.

$$S^k_i S^l_j G_{kl} = B_{ij}. \quad (3.2.4)$$

Already, we gave the equation of free-fall of the material-point. Similarly, the equation of the light-ray $x^i(\tau)$ is also given by (3.1.2). On the
other hand, the light-ray has to meet the equation (3.2.3) at all points. Therefore, we have
\[ 0 = \frac{d}{d\tau}(G_{ij}v^iv^j) = v^k\nabla_k(G_{ij}v^iv^j)v^k \]
\[ = (v^kG_{ij})v^iv^jv^k + 2G_{ij}(v^k\nabla_kv^i)v^j, \quad v^i = \frac{dx^i}{d\tau}. \quad (3.2.5) \]
By setting
\[ (v^k\nabla_kv^i)v^k = y[x^i/\tau] = 0, \]
we obtain
\[ (v^k\nabla_kG_{ij})v^iv^jv^k = 0. \quad (3.2.6) \]
The equation (3.2.6) has to apply to all the light-rays starting from \( P \). Therefore, the polynomial \( (v^kG_{ij})X^iX^jX^k \) can just be divided by the polynomial \( G_{ij}X^iX^j \), because \( G_{ij}X^iX^j \) is irreducible by Lemma 3.2.1 \((\rightarrow \text{reference}[2])\). Therefore \( 2A_i \) exists as follows.
\[ y^kG_{ij}X^iX^jX^k = (2A_iX^i)(G_{jk}X^jX^k). \quad (3.2.7) \]
Now, we pay attention to the \( A_i \). Let us change the light-cone from \( G_{ij} \) to \( \tilde{G}_{ij} = \lambda G_{ij} \). By the equation
\[ y^k\nabla_k\tilde{G}_{ij} = (\partial_k\lambda)G_{ij} + \lambda y^k\nabla_kG_{ij}, \quad (3.2.8) \]
we have
\[ y^k\nabla_k\tilde{G}_{ij}X^iX^jX^k = (\partial_k\lambda)G_{ij}X^iX^jX^k + \lambda y^k\nabla_kG_{ij}X^iX^jX^k. \quad (3.2.9) \]
By setting (3.2.7) to (3.2.9), we have
\[ y^k\nabla_k\tilde{G}_{ij}X^iX^jX^k = \{(\partial_k\lambda)G_{ij} + 2\lambda A_kG_{ij}\}X^iX^jX^k \]
\[ = 2\left(\frac{1}{2\lambda}\partial_k\lambda + A_k\right)\tilde{G}_{ij}X^iX^jX^k. \quad (3.2.10) \]
By the (3.2.10), we obtain
\[ \tilde{A}_k = A_k + \partial_k \log \sqrt{\lambda}. \quad (3.2.11) \]
Here, \( \tilde{A}_i \) is corresponding to \( \tilde{G}_{ij} \). By the equation (3.2.11), it seems that \( A_i \) is the vector potential of electromagnetism.
Lemma 3.2.1
If $G_{ij}$ is a light-cone, the polynomial $G_{ij}X^iX^j$ is irreducible.

(proof) We will lead a contradiction from the supposition which $G_{ij}X^iX^j$ is reducible. By a certain non-singular matrix $S_i^j$, we have

$$B_{ij} = S_i^k S_j^l G_{kl}. \quad (1)$$

If $G_{ij}X^iX^j$ is reducible, $a_i$ and $b_i$ exist as follows.

$$G_{ij}X^iX^j = a_iX^ib_jX^j. \quad (2)$$

Therefore we have

$$G_{ij} = \frac{1}{2}(a_ib_j + a_jb_i). \quad (3)$$

By using (1) and (3), we have

$$B_{ij} = \frac{1}{2}S_i^k S_j^l (a_kb_l + a_lb_k) = \frac{1}{2}(\tilde{a}_i\tilde{b}_j + \tilde{a}_j\tilde{b}_i). \quad (4)$$

Here,

$$\tilde{a}_i = S_i^p a_p, \quad \tilde{b}_i = S_i^p b_p. \quad (5)$$

In the special case of (4), we have

$$-1 = B_{11} = \tilde{a}_1\tilde{b}_1, \quad -1 = B_{22} = \tilde{a}_2\tilde{b}_2. \quad (6)$$

Therefore we have

$$\tilde{b}_1 = -\frac{1}{\tilde{a}_1}, \quad \tilde{b}_2 = -\frac{1}{\tilde{a}_2}. \quad (7)$$

Similarly by using (4), we have

$$0 = B_{12} = \frac{1}{2}(\tilde{a}_1\tilde{b}_2 + \tilde{a}_2\tilde{b}_1). \quad (8)$$

By setting (7) to (8), we have

$$0 = -\frac{1}{2}(\frac{\tilde{a}_1}{\tilde{a}_2} + \frac{\tilde{a}_2}{\tilde{a}_1}). \quad (9)$$

Multiplication by $\tilde{a}_1\tilde{a}_2$ to (9), we have

$$0 = \tilde{a}_1\tilde{a}_1 + \tilde{a}_2\tilde{a}_2. \quad (10)$$

We obtain $a_1 = a_2 = 0$ by (10), however these results contradict (6). □
3.3 Space-time-potential and gauge transformations.

Suppose that the light-cone $G_{ij}$ and the point-coordinate-system $(y^i)$ expressing the inertial frame of reference are given in the 4-space. Let $x^i(\tau)$ be the curve of free-fall of the material-point. Let $s$ be the arc-length measured with the metric $G_{ij}$ along this curve, i.e.,

$$ds^2 = G_{ij}dx^i dx^j.$$  \hspace{0.5cm} (3.3.1)

According to Lemma 2.4.1

$$\frac{d^2 \tau}{ds^2} + \frac{1}{2}(\nabla_k G_{ij})V^i V^j V^k \frac{d\tau}{ds} = 0 \quad , \quad V^i = \frac{dx^i}{ds}. \hspace{0.5cm} (3.3.2)$$

On the other hand, according to the section 3.2,

$$(\nabla_k G_{ij})V^i V^j V^k = 2(A_k V^k)(G_{ij} V^i V^j). \hspace{0.5cm} (3.3.3)$$

Because $G_{ij} V^i V^j = 1$, we obtain

$$\frac{d^2 \tau}{ds^2} + (A_k V^k) \frac{d\tau}{ds} = 0. \hspace{0.5cm} (3.3.4)$$

Let $P, Q$ be two point on the $x^i(\tau)$. We consider

$$\zeta(P) = - \int_Q^P A_i dx^i + C. \hspace{0.5cm} (3.3.5)$$

Here, $C$ is a constant. If $\tau$ is defined as

$$d\tau = \exp(\zeta) ds, \hspace{0.5cm} (3.3.6)$$

then

$$\frac{d^2 \tau}{ds^2} = \exp(\zeta) \frac{d\zeta}{ds} = - \exp(\zeta) A_i \frac{dx^i}{ds}. \hspace{0.5cm} (3.3.7)$$

The equation (3.3.7) shows that $\tau$ is the solution of the equation (3.3.4).

In this paper, $\zeta$ is called a space-time-potential.

By (3.3.6),

$$d\tau^2 = \exp(2\zeta) G_{ij} dx^i dx^j. \hspace{0.5cm} (3.3.8)$$

We hope to deal with $\exp(2\zeta) G_{ij}$ as the metric, however $\zeta$ is not a function in the 4-space $(x^i)$. Then, let us extend the space-time to a 5-space $(x^\lambda)$, and let us consider $x^0 = \zeta$. We define a new metric $g_{\lambda\mu}$ in the 5-space $(x^\lambda)$ as follows.

$$g_{ij} = \exp(2x^0) G_{ij}(x^1, ..., x^4), \quad g_{0\lambda} = g_{0\lambda} = 0. \hspace{0.5cm} (3.3.9)$$
According to the definitions, the curve $x^i(\tau)$ is written $x^\lambda(\tau)$ in the 5-space $(x^\lambda)$. Let $dx^\lambda$ be a line element on this curve. Then,

$$dx^0 = d\zeta = -A_i dx^i, \quad (3.3.10)$$

i.e.,

$$dx^0 + A_i dx^i = 0. \quad (3.3.11)$$

If we define $A_0 = 1$ as a fifth element of $A_i$, then we can write (3.3.11) as follows.

$$A_\lambda dx^\lambda = 0. \quad (3.3.12)$$

In this paper, transformations appeared by $G_{ij} \rightarrow \lambda G_{ij}$ are called a gauge transformation. As an example, we have

$$A_i \rightarrow A_i + \partial_i \eta, \quad \eta = \log \sqrt{\lambda}. \quad (3.3.13)$$

How does the space-time-potential of the curve transform by the gauge transformation? Let $\tilde{\xi}$ be a space-time-potential of the new gauge. According to the definitions,

$$d\tilde{\xi} = -(A_i + \partial_i \eta)dx^i, \quad \tilde{\xi}(P) = -\int_Q^P d\tilde{\xi} + C. \quad (3.3.14)$$

Here, $Q$ and $C$ are not fixed. Then, let us suppose that the proper-time does not vary by the gauge transformation. That is,

$$ds^2 = \exp(2\zeta)G_{ij}dx^idx^j = \exp(2\tilde{\xi})\lambda G_{ij}dx^idx^j = \exp(2\tilde{\xi} + 2\eta)G_{ij}dx^idx^j. \quad (3.3.15)$$

Therefore

$$\zeta(P) = \tilde{\xi}(P) + \eta(P). \quad (3.3.16)$$

Now, we consider the transformation of coordinates as follows.

$$\bar{x}^0 = x^0 - \eta(x^1, \ldots, x^4), \quad \bar{x}^i = x^i. \quad (3.3.17)$$

By (3.3.17), $A_\lambda$ transform as follows.

$$\bar{A}_0 = \frac{\partial x^0}{\partial \bar{x}^0} A_0 + \frac{\partial x^0}{\partial \bar{x}^0} A_j = 1 + \delta_0^j A_j = 1, \quad (3.3.18)$$

$$\bar{A}_i = \frac{\partial x^0}{\partial \bar{x}^i} A_0 + \frac{\partial x^0}{\partial \bar{x}^i} A_j = \partial_i \eta + \delta_i^j A_j = A_i + \partial_i \eta. \quad (3.3.19)$$
Generally by using (3.3.17), a symmetric tensor $c_{\lambda\mu}$ of type (0,2) transform as follows.

$$
\tilde{c}_{ij} = c_{ij} + \partial_i \eta c_{0j} + \partial_j \eta c_{0i} + \partial_i \eta \partial_j \eta c_{00},
$$

$$
\tilde{c}_{0j} = c_{0j} + \partial_j \eta c_{00}, \quad \tilde{c}_{00} = c_{00}. \quad (3.3.20)
$$

In the case of $g_{\lambda\mu}$, we have

$$
\tilde{g}_{ij} = g_{ij}, \quad \tilde{g}_{0\lambda} = \tilde{g}_{\lambda 0} = 0. \quad (3.3.21)
$$

### 3.4 Metrics of 5-space.

The metric $g_{\lambda\nu}$ defined in section 3.3 has not a inverse matrix. If $g_{\lambda\nu}$ has a inverse matrix $g^{\lambda\nu}$ then $g^{\lambda\nu} g_{\nu\mu} = \delta_\mu^\lambda$. In the case of $\lambda = \mu = 0$,

$$
0 = g^{\nu\nu} g_{\nu 0} = \delta_0^0 = 1.
$$

This is a contradiction. Therefore, $g_{\lambda\nu}$ is abnormal as the metric of the 5-space. Let us define a normal metric $h_{\lambda\mu}$ extended $g_{\lambda\mu}$.

If a vector $V^\lambda$ grown from a point $P$ is $A_{\lambda}(P)V^\lambda = 0$ then we wish

$$
h_{\lambda\mu}(P)V^\lambda V^\mu = g_{\lambda\mu}(P)V^\lambda V^\mu. \quad (3.4.1)
$$

Therefore, the polynomial

$$
(h_{\lambda\mu} - g_{\lambda\mu})X^\lambda X^\mu \quad (3.4.2)
$$

can just be divided by the polynomial $A_{\mu}X^\mu$. We can find out $a_\lambda$ as follows.

$$
(h_{\lambda\mu} - g_{\lambda\mu})X^\lambda X^\mu = (a_\lambda X^\lambda)(A_{\mu}X^\mu). \quad (3.4.3)
$$

As a result, we obtain

$$
h_{\lambda\mu} = g_{\lambda\mu} + \frac{1}{2}(a_\lambda A_{\mu} + a_{\mu} A_{\lambda}). \quad (3.4.4)
$$

By (3.3.20), the metric $h_{\lambda\mu}$ transforms as follows.

$$
\tilde{h}_{ij} = h_{ij} + \partial_i \eta h_{0j} + \partial_j \eta h_{0i} + \partial_i \eta \partial_j \eta h_{00}, \quad (3.4.5)
$$

$$
\tilde{h}_{0j} = h_{0j} + \partial_j \eta h_{00}, \quad \tilde{h}_{00} = h_{00}. \quad (3.4.6)
$$

In (3.4.6), we know that $h_{0j}/h_{00}$ has the same transformation as $A_i$.

Therefore, let us define the following.

$$
h_{0j} = h_{00} A_j. \quad (3.4.7)$$
By using (3.4.4),

$$h_{00} = a_0. \quad (3.4.8)$$

By using (3.4.7) and (3.4.8),

$$h_{0j} = a_0 A_j. \quad (3.4.9)$$

On the other hand, by using (3.4.4)

$$h_{0j} = \frac{1}{2}(a_0 A_j + a_j). \quad (3.4.10)$$

By using (3.4.10) and (3.4.9)

$$a_j = a_0 A_j.$$  

On the other hand $a_0 = a_0 A_0$, therefore $a_\lambda = a_0 A_\lambda$. As a result, we obtain

$$h_{\lambda \mu} = g_{\lambda \mu} + a_0 A_\lambda A_\mu. \quad (3.4.11)$$

Lastly, we have to decide $a_0$. Let us consider $dx^\lambda = (dx^0, 0, 0, 0)$. The length of $dx^\lambda$ is

$$dl^2 = h_{\lambda \mu} dx^\lambda dx^\mu = h_{00} dx^0 dx^0 = a_0 dx^0 dx^0. \quad (3.4.12)$$

We will expect $dl^2 = dx^0 dx^0$, i.e., $a_0 = 1$. We obtain

$$h_{\lambda \mu} = \exp(2x^0) G_{\lambda \mu} + A_\lambda A_\mu. \quad (3.4.13)$$

If we disregard $\exp(2x^0)$, $h_{\lambda \mu}$ is same as the Kaluza's metric.

The $h_{\lambda \mu}$ has an inverse matrix $h^{\lambda \mu}$ as follows.

$$h^{ij} = g^{ij}, \quad h^{i0} = h^{0i} = -g^{ij} A_j,$$

$$h^{00} = g^{ij} A_i A_j + 1, \quad g^{ij} = \exp(-2x^0) G^{ij}.$$ 

References
